

## Effective descriptions of branes on non-geometric tori

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ABSTRACT: We investigate the low-energy effective description of non-geometric compactifications constructed by T-dualizing two or three of the directions of a  $T^3$  with non-vanishing  $H$ -flux. Our approach is to introduce a D3-brane in these geometries and to take an appropriate decoupling limit. In the case of two T-dualities, we find at low energies a non-commutative  $T^2$  fibered non-trivially over an  $S^1$ . In the UV this theory is still decoupled from gravity, but is dual to a little string theory with flavor. For the case of three T-dualities, we do not find a sensible decoupling limit, casting doubt on this geometry as a low-energy effective notion in critical string theory. However, by studying a topological toy model in this background, we find a non-associative geometry similar to one found by Bouwknegt, Hannabuss, and Mathai.

KEYWORDS: Non-Commutative Geometry, Flux compactifications.

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## 1. Introduction

Compactifications on non-geometric spaces have emerged as a new potential class of string theory vacua [1–3]. The non-geometric nature of these spaces arises because some of the transformations that glue the patches together include U-dualities as well as the standard diffeomorphisms. A useful prototype non-geometric space is found by T-dualizing a three dimensional torus with a non-vanishing  $H$ -flux: T-dualizing on one cycle gives rise to a Scherk-Schwarz twisted torus [4] which is purely geometric. However, T-dualizing on two cycles gives rise to a space in which one of the cycles is periodic up to T-duality, which mixes momentum and winding modes. As such, in this space, geometric notions such as the metric and the background  $B$ -field are only well-defined locally.

After T-dualizing twice, one can contemplate T-dualizing along the third direction of the  $T^3$ . Naively, a  $T^3$  with uniform  $H$ -flux is isometric under shifts in three independent dimensions. However, the Buscher rules for T-duality require that the two-form potential  $B$  be uniform [5, 6], and in order for the three form field strength  $H$  to be uniform, the two form potential must break at least one of the translation isometries. Thus, the standard Buscher rules do not apply.

Recently, Shelton, Taylor, and Wecht proposed an interpretation of the third T-dual of a torus with  $H$ -flux as an example of a consistent non-geometric compactification [7]. They also introduced a nomenclature which we follow: the  $H$ ,  $f$ ,  $Q$ , and  $R$ -spaces correspond to a  $T^3$  with  $H$ -flux T-dualized zero, one, two, and three times, respectively. The  $Q$ -space is perhaps the simplest example of a non-geometric compactification that mixes momentum and winding modes. If the interpretation of [7] is correct, the  $R$ -space is an example of a space that is more non-geometric than the  $Q$ -space.<sup>1</sup>

Space-time geometry in string theory is an approximate notion which is valid only when the scale of all the geometric features are much larger than string scale. When the size of a compact manifold becomes comparable to the string scale, geometric notions break down due to the non-locality intrinsic to the fact that a string is an extended object. In order to isolate the novel geometric features from the generic non-locality effects of string theory, one takes the decoupling limit,  $\alpha' \rightarrow 0$ , keeping the size of the compact manifold finite. Geometric notions acquire precise meaning in this limit. However, not all space-time geometries one obtains in this way are ordinary geometric spaces. More exotic spaces, such as the non-commutative plane, can arise as decoupling limits of string theory.

One can investigate the geometric features of the  $Q$ -space and the  $R$ -space along similar lines. Our strategy is to use the properties of field theories defined on these spaces as a probe of the geometry. In string theory, this can be implemented by introducing a D3-brane filling the space and taking the decoupling limit.<sup>2</sup> This gives rise to a non-trivial effective dynamics of open strings ending on a D-brane in a presence of an NSNS  $B$ -field background.

Because of the presence of the  $B$ -field, it is natural to expect some connection between the  $Q/R$ -spaces and non-commutative geometry. Indeed, in the case of the  $Q$ -space, we find a familiar non-commutative theory whose UV completion is a little string theory coupled with flavor. On the other hand, we do not find a clean decoupling limit for a theory defined on  $R$ -space. The  $R$ -space does not appear to admit an effective description as a smooth macroscopic structure decoupled from gravity.

This article is organized as follows. In section 2 we review the supergravity background giving rise to the  $H$ ,  $f$ , and  $Q$  spaces. In section 3, we describe the decoupled theory of D3-branes in the  $Q$ -space and its UV completion. In section 4, we comment on the status of  $R$ -space. In section 5 we discuss a toy model for  $R$ -space physics. We end with some concluding remarks in section 6.

## 2. Torus with $H$ -flux and its T-duals

In this section, we review the spaces  $H$ ,  $f$ ,  $Q$ , and  $R$ , in the nomenclature of [7]. We will also review the warped supergravity background of smeared NS5-branes that properly takes into account the gravitational backreaction of the non-vanishing  $H$ -flux.

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<sup>1</sup>These fluxes are also interesting from the “cosmological billiards” perspective, since they correspond to interesting roots of  $E_{10}$  [8, 9]. We thank Ori Ganor for bring this point to our attention.

<sup>2</sup>Adding D-branes to various T-duals of  $H$ -space was also studied in [10, 11].

### 2.1 The $H$ -space

Consider a  $T^3$  with  $H$ -flux whose coordinates are given by  $x_i$  with periodicities  $L_i$  for  $i = 1, 2, 3$ . When written explicitly, the background  $B$ -field breaks at least one of the infinitesimal translation symmetries:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2, \quad B = b x_1 dx_2 \wedge dx_3, \quad H = dB, \quad b = \frac{(2\pi)^2 N \alpha'}{L_1 L_2 L_3}. \quad (2.1)$$

The  $H$ -flux is quantized so that  $N$  is an integer. We refer to this space as the  $H$ -space using a notation similar to [7].

### 2.2 The $f$ -space

T-dualizing along  $x_3$  gives rise to a purely metrical background

$$ds^2 = dx_1^2 + dx_2^2 + (d\tilde{x}_3 + b x_1 dx_2)^2, \quad (2.2)$$

with twisted boundary conditions,

$$x_1 \sim x_1 + n_1 L_1, \quad x_2 \sim x_2 + n_2 L_2, \quad \tilde{x}_3 \sim \tilde{x}_3 + n_3 \tilde{L}_3 - n_1 b L_1 x_2, \quad \tilde{L}_3 = \frac{(2\pi)^2 \alpha'}{L_3}. \quad (2.3)$$

This space is also known as the *nil manifold* or the *twisted torus* and is topologically distinct from the ordinary torus; for example, it has  $H_1(\mathbf{Z})$  given by  $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_N$  [3, 11]. Following [7] we refer to it as the  $f$ -space.

Introducing dimensionless coordinates,

$$x_i = L_i y_i, \quad (2.4)$$

the metric and the boundary condition becomes more transparent:

$$ds^2 = L_1^2 dy_1^2 + L_2^2 dy_2^2 + L_3^2 (d\tilde{y}_3 + N y_1 dy_2)^2, \quad (2.5)$$

$$y_1 \sim y_1 + n_1, \quad y_2 \sim y_2 + n_2, \quad \tilde{y}_3 \sim \tilde{y}_3 + n_3 - n_1 N y_2. \quad (2.6)$$

In this form, it is also apparent that  $N$  is quantized to be an integer.

### 2.3 The $Q$ -space

Further T-dualizing along the  $x_2$  direction gives the background,

$$\begin{aligned} ds^2 &= dx_1^2 + \frac{1}{1 + b^2 x_1^2} (d\tilde{x}_2^2 + d\tilde{x}_3^2), \\ B &= \frac{b x_1}{1 + b^2 x_1^2} d\tilde{x}_2 \wedge d\tilde{x}_3, \\ e^{\phi - \phi_0} &= \frac{1}{\sqrt{1 + b^2 x_1^2}}. \end{aligned} \quad (2.7)$$

This space is called the  $Q$ -space in [7]. This background is periodic in the  $x_1$  direction up to a T-duality of the  $\tilde{x}_2$ - $\tilde{x}_3$  torus, which exchanges momentum and winding modes. As is clear from (2.7), the metric and  $B$ -field are locally defined, but globally are not manifestly periodic in the  $x_1$  direction. As such we take it as a prototypical example of a “non-geometric” space [1–3].

## 2.4 The $R$ -space

It is not completely clear that the  $Q$ -space can be further T-dualized along the  $x_1$  coordinate. Translation along the  $x_1$  direction is, after all, not (even locally) an isometry. Assuming that a T-dual does exist, this space was named the  $R$ -space by [7].

## 2.5 Smearred NS5-brane background

As is, the  $T^3$  with non-vanishing  $H$ -flux described in section 2.1 is not a consistent closed string background since it does not take into account the gravitational backreaction of the  $H$ -flux. One convenient way to build a consistent background is to start with the NS5-brane background (extended along the 56789 directions),

$$\begin{aligned} ds^2 &= -dt^2 + dx_5^2 + \dots + dx_9^2 + f(r)(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \\ H &= *(dt \wedge dx_5 \wedge \dots \wedge dx_9 \wedge df^{-1}), \\ e^\phi &= g_s f(r)^{1/2}, \\ f(r) &= 1 + \frac{m\alpha'}{r^2}, \end{aligned} \tag{2.8}$$

which is a magnetic source of the 3-form  $H$ -field, and to smear it along three of the four transverse coordinates so that the supergravity background becomes

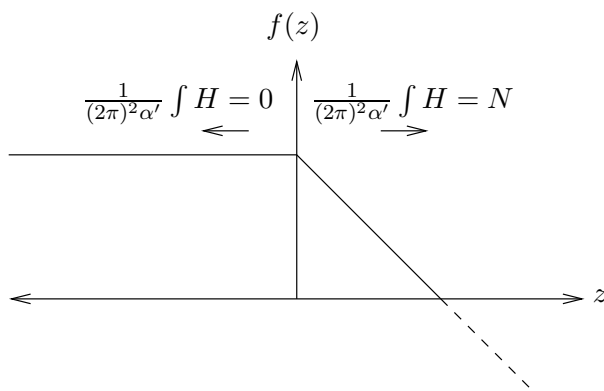
$$\begin{aligned} ds^2 &= -dt^2 + dx_5^2 + \dots + dx_9^2 + f(r)(dx_1^2 + dx_2^2 + dx_3^2 + dz^2), \\ H &= dB, \\ B &= \frac{(2\pi)^2 N\alpha'}{L_1 L_2 L_3} (x_1 - x_1^0) dx_2 \wedge dx_3, \\ e^\phi &= g_s f(z)^{1/2}, \\ f(z) &= f_0 - \frac{(2\pi)^2 N\alpha' (|z| + z)}{2L_1 L_2 L_3}. \end{aligned} \tag{2.9}$$

The smeared NS5-branes are located at  $z = 0$ . The parameter  $f_0$ , which we take to be positive, is otherwise freely adjustable. The parameter  $x_1^0$  is just an additive constant for the  $x_1$  coordinate. We have tuned the charges at infinity so that  $H = 0$  for  $z < 0$ . The smeared NS5-brane acts as a domain wall source for a uniform three-form field strength  $H$  in the region  $z > 0$ . See figure 1 for an illustration.

This is not the only way to construct a solution to the supergravity equations of motion with non-vanishing  $H$ -flux threading a  $T^3$ . This solution, however, is convenient in that it preserves 16 of the 32 supersymmetries in type IIA or type IIB supergravity. We will consider T-duals of (2.9) in order to consistently embed the  $H$ ,  $f$ ,  $Q$ , and  $R$  spaces into type IIA/B supergravity.

## 3. Decoupled theory of D3-branes in $Q$ -space

In this section, we derive an effective geometry for D3-branes wrapping the  $Q$ -space. This setup is equivalent to starting with a D1-brane wrapping the  $x_1$  direction of the  $H$ -space



**Figure 1:** Profile of the harmonic function  $f(z)$ .

and T-dualizing along the  $x_2$  and  $x_3$  directions;<sup>3</sup>

	0	1	2	3	4	5	6	7	8	9	
NS5	•	≡	≡	≡		•	•	•	•	•	(3.1)
D1	•	•									

This configuration preserves 8 of the 32 supersymmetries of type IIB, as can be seen by T-dualizing along the  $x_5$  and the  $x_6$  coordinates, S-dualizing, and counting the number of relatively transverse coordinates of the branes.

Since the  $H$ -space metric (2.9) is manifestly isometric along the  $x_2$  and  $x_3$  directions, it is straightforward to T-dualize along these coordinates, giving the background,

$$\begin{aligned}
 ds^2 &= f(U)dx_1^2 + \frac{f(U)}{f(U)^2 + \left(\frac{N\tilde{L}_2\tilde{L}_3}{\alpha'L_1}(x_1 - x_1^0)\right)^2}(d\tilde{x}_2^2 + d\tilde{x}_3^2), \\
 B_{23} &= \frac{\frac{N\tilde{L}_2\tilde{L}_3}{\alpha'L_1}(x_1 - x_1^0)}{f(U)^2 + \left(\frac{N\tilde{L}_2\tilde{L}_3}{\alpha'L_1}(x_1 - x_1^0)\right)^2}, \\
 f(U) &= f_0 - \frac{N\tilde{L}_2\tilde{L}_3(|U| + U)}{2L_1},
 \end{aligned}
 \tag{3.2}$$

where

$$\tilde{L}_{2,3} = (2\pi)^2 \frac{\alpha'}{L_{2,3}}
 \tag{3.3}$$

is the period of the dual coordinates  $\tilde{x}_{2,3}$ . The transverse coordinate  $x_4$  has been scaled as

$$x_4 = \alpha' U
 \tag{3.4}$$

so that  $U$  parameterizes the vacuum expectation value of a scalar field polarized along the  $x_4$  direction.

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<sup>3</sup>We use a • to denote directions along which a brane is extended and ≡ to denote directions along which it is smeared.

In order to take the decoupling limit, we send  $\alpha' \rightarrow 0$  keeping the field theory parameters  $L_1$ ,  $\tilde{L}_{2,3}$  and  $U$  fixed. In this limit, the metric and the  $B$ -field degenerate; however, since we are interested in the effective dynamics of decoupled open strings, we should not be concerned about the scaling of the closed string variables  $g$  and  $B$ . Instead, we should consider the open string metric and non-commutativity parameter along the lines of [12–16];

$$(g + B)^{-1} = G + \frac{\theta}{2\pi\alpha'}. \tag{3.5}$$

Explicitly, we find

$$G^{ij} = f(U) \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{ij}, \quad \Theta^{23} = \frac{2\pi\theta^{23}}{\tilde{L}_2\tilde{L}_3} = N \left( \frac{x_1 - x_1^0}{L_1} \right). \tag{3.6}$$

The dimensionless non-commutativity parameter  $\Theta^{23}$  is simply the non-commutativity parameter  $2\pi\theta^{23}$  divided by the volume of the torus  $\tilde{L}_2\tilde{L}_3$ . Rather remarkably, both the open string metric and the non-commutativity parameters remain finite in the limit.

The manifestation of the non-geometric character of the  $Q$ -space in the decoupled theory is now transparent. Since the non-commutativity parameter  $\Theta^{23}$  depends explicitly on  $x_1$ , one cannot naively make this coordinate periodic. However, under a discrete shift  $x_1 \rightarrow x_1 + L_1$ , the dimensionless non-commutativity parameter  $\Theta^{23}$  shifts by  $N$ . Such a shift of  $\Theta^{23}$  by an integer is an example of Morita equivalence, which acts as an  $SL(2, Z)$  transformation on the parameters of non-commutative torus (using the notation explained in the appendix of [17]) as follows:

$$\begin{aligned} \tilde{\Theta} &= \frac{c + d\Theta}{a + b\Theta}, & \tilde{\Phi} &= (a + b\Theta)^2\Phi - b(a + b\Theta), & \tilde{\Sigma} &= (a + b\Theta)\Sigma, \\ \tilde{g}_{\text{YM}}^2 &= (a + b\Theta)g_{\text{YM}}^2, & \begin{pmatrix} \tilde{m} \\ \tilde{N} \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ N \end{pmatrix}. \end{aligned} \tag{3.7}$$

Therefore, in the  $Q$ -space, the  $x_1$  coordinate is only periodic up to a Morita transformation. Morita equivalence is precisely the structure inherited from T-duality in the decoupling limit. It is therefore natural to find a compactification that identifies shifts in  $x_1$  via Morita equivalence emerging as a decoupling limit of a compactification that identifies shifts in  $x_1$  via T-duality.

This geometrical structure, which can be viewed as a field of non-commutative tori fibered over a circle, also appears in the works of [18–21]. In our work, we emphasize the fact that this structure has a physical origin as the decoupled theory of the open string excitations living on the world volume of a D3-brane embedded into the  $Q$ -space.

Although the motivation was somewhat different, most of the features of the decoupled field theory on D3-branes in the  $Q$ -space were first worked out in [22]. One feature, which did not get emphasized in [22], however, is that the open-string metric (3.6) is warped in the transverse coordinate  $U$ . In fact, there will be a singularity at some finite value of  $U$ . Since a typical string fluctuates by a size of the order of  $l_s$ , which is much larger than the

distance to the singularity, which is order  $l_s^2$ , the open string dynamics is strongly influenced by the presence of the singularity. Hence, it would be premature to assume, for example, that the low-energy effective theory is precisely  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory non-commutated by the position dependent  $\Theta$  given in (3.6). Such a theory would have an unbroken  $SO(6)$  R-symmetry group, which is clearly broken by the warping along the  $U$  direction.

Since there is no physical scale other than the scale of compactification and the non-commutativity, one can also think of this system as having non-commutativity parameters that are different for different  $U(1)$  subsectors when the gauge group is broken by turning on the vacuum expectation values for the transverse scalar field along the warped coordinate  $U$ . Models with these features have been considered before [23, 24]. The authors of [24] referred to these models as “non-abelian geometry.”

In order to infer the correct low-energy effective action for the decoupled theory, one must analyze the dynamics of open strings in this background in some detail.<sup>4</sup> This seems like a serious technical challenge in light of the fact that the world sheet sigma model for strings propagating in the background of smeared NS5-branes does not appear to be exactly solvable.<sup>5</sup>

### 3.1 UV completion of the $Q$ -space effective geometry

That there are singularities at a finite distance in moduli-space suggests that the effective description based on non-commutative field theory is breaking down because of certain states that were integrated out. In the remainder of this section, we will work out a UV completion of this theory that resolves the singularity while keeping gravity decoupled.

The singularity of the smeared NS5-brane background is closely related to the singularity in type I' theory that one encounters in the heterotic/type I duality [26]. The mechanism that resolves the singularity is also similar. To see this more explicitly, it is useful to embed the 1+1 dimensional effective dynamics of D1-branes in the configuration (3.1) as a dimensional reduction of 3+1 dimensional system oriented as follows:

*Step I:*

$$\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 \text{NS5} & \bullet & \equiv & \equiv & \equiv & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 \text{D3} & \bullet & \bullet & & & & \bullet & \bullet & & & 
 \end{array} \tag{3.8}$$

The coordinates  $x_5$  and  $x_6$  are taken to be compact with period  $L_{5,6}$ , which will remain finite in the scaling limit. ( $L_{5,6}$  can be taken to be small compared to other scales of the problem at the very end.) In order for this background to yield the  $Q$ -space after two T-dualities, we scale the parameters of the compactification as follows:<sup>6</sup>

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<sup>4</sup>The authors of [24] proposed a generic non-abelian  $\ast$ -product, but we do not see how such a product properly incorporates the dynamics of open strings in this background.

<sup>5</sup>In order to study explicit realization of non-abelian geometry, one can instead consider simpler construction based on Melvin universes which are solvable [25].

<sup>6</sup>Factors of 2 and  $\pi$  are left out to prevent cluttering.



$$g_I = g, \quad \alpha'_I = \alpha', \quad L_{1I} = L_1, \quad L_{2,3I} = \frac{\alpha'}{\tilde{L}_{23}}. \quad (3.9)$$

*Step II: ST<sub>123</sub>-duality* Next, we S-dualize and then perform T-duality along the  $x_1$ ,  $x_2$ , and  $x_3$  directions. This gives the configuration,

	0	1	2	3	4	5	6	7	8	9	
D8	•	•	•	•		•	•	•	•	•	(3.10)
D4	•		•	•		•	•				

where the parameters after the duality scale as

$$g_{II} = \frac{gL_2L_3}{L_1\alpha'_{II}{}^{1/2}}, \quad \alpha'_{II} = g\alpha', \quad L_{1II} = \frac{g\alpha'}{L_1}, \quad L_{2,3II} = g\tilde{L}_{2,3}. \quad (3.11)$$

This is the same brane configuration discussed in [27–29] that gives rise to non-trivial fixed points in 4+1 dimensions precisely when the D4-brane is placed at the singularity. The singularity in this context arises from integrating out the strings stretching between the D4-brane and the D8-brane. It is convenient to view this system as a decompactification limit of a type I' theory given by separating the D8-branes from the O8-branes. To match the harmonic function profile with what is illustrated in figure 1, one should imagine eight D8-branes and an O8 brane to the far left,  $N$  D8-branes at  $z = 0$ , and  $(8 - N)$  D8-branes and an O8-brane to the far right.

*Step III: 1-11 flip*

The only thing which makes this description unsuitable as an effective description of the  $Q$ -space is the small size of the period of the  $x_1$  direction. This can be rectified by performing a 1-11 flip, which yields the configuration,

	0	1	2	3	4	5	6	7	8	9	
D8	•	•	•	•		•	•	•	•	•	(3.12)
NS5	•	•	•	•		•	•				

with parameters

$$g_{III} = \frac{\alpha'\sqrt{g}}{L_1\sqrt{\tilde{L}_2\tilde{L}_3}}, \quad L_{1III} = \frac{g\tilde{L}_2\tilde{L}_3}{L_1}, \quad \alpha'_{III} = g\tilde{L}_2\tilde{L}_3, \quad L_{2,3III} = g\tilde{L}_{2,3}. \quad (3.13)$$

Notice that as  $\alpha' \rightarrow 0$ ,  $g_{III}$  goes to zero as well, while  $\alpha'_{III}$  and the various length scales associated with the world volume of the NS5-brane remain fixed. This is the standard decoupling limit of little string theory.

An important set of light degrees of freedom in this limit are the D2-branes stretching between the NS5-brane and the D8-branes. In the limit, the D2-brane behaves

effectively as a string. If the distance separating the NS5-brane and the D8/O8-brane is of the order  $\alpha'U$ , then the tension of this effective string is

$$T = \frac{1}{g_{III}\alpha'_{III}{}^{3/2}}\alpha'U = \frac{L_1U}{g^2\tilde{L}_2\tilde{L}_3}, \tag{3.14}$$

which remains finite as  $\alpha' \rightarrow 0$ . These states are the analogues of “fundamental matter” for little string theory, along the lines discussed for the case of D3-branes in [30]. It is also the IIA reduction (along one of the dimensions transverse to the M5-brane) of the non-critical string theory introduced in [31]. The conclusion is that gauge theory on  $Q$ -space is a low-energy effective description of “little string theory with flavor,” which is decoupled from gravity.

#### 4. Low-energy effective description of the $R$ -space

We will now describe what happens if one attempts to repeat the story for the  $R$  space. That we do not have an explicit supergravity solution describing the  $R$ -space analogous to (3.2) prevents us from directly inserting a D3-brane as we did in the previous section. What we can do instead is to insert a D0-brane in the  $H$ -space, and study the low-energy effective dynamics, while scaling the size of the torus as  $L_{1,2,3} \sim \alpha'/\tilde{L}_{1,2,3}$ . This scaling isolates the winding modes while decoupling the momentum modes as  $\alpha' \rightarrow 0$ . Re-interpreting the winding modes of one geometry as Kaluza-Klein excitations of some other geometry essentially amounts to performing a T-duality.<sup>7</sup> In fact, had the  $B$ -field been taken to be a constant, this is precisely the approach taken in [34–36] to construct ordinary non-commutative spaces as a decoupling limit.

One can easily check that D0-branes in the background of smeared NS5-branes,

$$\begin{array}{cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{NS5} & \bullet & \equiv & \equiv & \equiv & & \bullet & \bullet & \bullet & \bullet & \bullet \\ \text{D0} & \bullet & & & & & & & & & \end{array} \tag{4.1}$$

break all supersymmetries and, hence, there will be a potential for the D0-brane to roll toward the NS5-branes. An alternative setup is to consider a D1-brane extended along the warped direction:

$$\begin{array}{cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{NS5} & \bullet & \equiv & \equiv & \equiv & & \bullet & \bullet & \bullet & \bullet & \bullet \\ \text{D1} & \bullet & & & & \bullet & & & & & \end{array} \tag{4.2}$$

This configuration is supersymmetric and static. In order to isolate the low-energy effective dynamics on the  $R$  space, we scale our parameters as

$$g_s = \frac{g_{YM}^2 M_4 \alpha'}{\tilde{L}_1 \tilde{L}_2 \tilde{L}_3}, \quad L_{123} = \frac{\alpha'}{\tilde{L}_{1,2,3}}, \tag{4.3}$$

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<sup>7</sup>One can in fact think of our construction as the generalization of [32, 33] on  $T^3$  with non-vanishing  $H$ -field.

so that the volume  $\tilde{L}_1\tilde{L}_2\tilde{L}_3$  and the gauge coupling  $g_{YM4}^2$  of the 4+1 dimensional Yang-Mills theory are finite after T-dualizing along  $x_1, x_2$ , and  $x_3$  coordinates.

There is a problem with interpreting such a construction as effectively giving rise to the  $R$ -space. The smeared NS5-brane background is still given by (2.9), but now the harmonic function  $f(z)$ , when written in terms of  $\alpha'$  and the variables that are kept finite in the scaling limit, takes the form

$$f(z) = f_0 - \frac{N\tilde{L}_1\tilde{L}_2\tilde{L}_3(|z| + z)}{2(2\pi)^4\alpha'^2} . \tag{4.4}$$

For the  $R$ -space to be a useful effective notion, one would like  $f(z)$  to be of order 1 for a sufficiently wide range of values of  $z$ . Clearly, this is not the case in the limit  $\alpha' \rightarrow 0$ .

Such severe warping appears to make the  $R$ -space problematic as a low-energy effective notion. The appearance of a strong gravitational backreaction is to be expected since we start by shrinking the volume of the torus in the  $H$ -space, where the total flux is kept fixed. As we shrink the torus, the energy density associated with the flux increases, giving rise to stronger backreaction on the geometry. Such a strong gravitational backreaction is also potentially problematic for the effective description of the  $Q$ -space considered in the previous section. It is the combination of the fact that the probe branes can be arranged to be localized in the warped direction, and that the amount of squeezing of the flux is milder in the scaling relevant to the  $Q$ -space that allows for a smooth decoupling limit.

In spite of the strong warping, one could contemplate performing a duality transformation on (4.2) to attempt to identify the analogue of (3.12) for the  $R$ -space case.

*Step I:  $T_{56}$  duality:*

	0	1	2	3	4	5	6	7	8	9	
NS5	•	≡	≡	≡		•	•	•	•	•	(4.5)
D3	•				•	•	•				

$$g_I = \frac{g_s L_5 L_6}{\alpha'} = \frac{g_{YM4}^2 L_5 L_6}{\tilde{L}_1 \tilde{L}_2 \tilde{L}_3} = \text{finite}, \quad L_{1,2,3I} = \frac{\alpha'}{\tilde{L}_{1,2,3}}, \quad L_{5,6I} = \text{finite} . \tag{4.6}$$

*Step II:  $ST_{123}$  duality:* This duality transforms (4.6) to

	0	1	2	3	4	5	6	7	8	9	
D8	•	•	•	•		•	•	•	•	•	(4.7)
D6	•	•	•	•	•	•	•				

with the parameters

$$g_{II} = \frac{g_I^2 \tilde{L}_1 \tilde{L}_2 \tilde{L}_3}{\alpha'_{II}{}^{3/2}}, \quad \alpha'_{II} = g_I \alpha'_I, \quad L_{1,2,3II} = g_I \tilde{L}_{1,2,3} . \tag{4.8}$$

This gives rise to a low-energy effective coupling for the D6-brane,

$$g_{YM6}^2 = g_{II} \alpha'_{II}{}^{3/2} = g_I^2 \tilde{L}_1 \tilde{L}_2 \tilde{L}_3 = \text{finite} . \tag{4.9}$$

In fact, this configuration is the infinite volume limit of the 6+1 dimensional gauge theory discussed in [37]. Unlike the NS5-branes considered in the previous section, D6-branes do not have a scaling limit that decouples the gauge dynamics from gravity. Nonetheless, it is tempting to speculate that the  $R$ -space emerges as a low-energy effective description of this dynamical system. Unfortunately, as the slope of the harmonic function (4.4) diverges in the  $\alpha' \rightarrow 0$  limit, the range of the coordinate  $z$  for which this system is dual to an  $R$ -space shrinks to zero size, making such an interpretation doubtful.

Had we considered instead the non-supersymmetric D0-brane configuration shown in (4.1), the warping would cause the D0-brane to roll toward the NS5-branes in a time-scale that gets arbitrarily short as  $\alpha'$  is sent to zero.

All of these problems are a consequence of generic gravitational backreaction effects arising from quantized fluxes in a small volume. While we demonstrated the difficulty only for the specific case of  $H$ -flux generated by the background of smeared NS5-branes, it seems reasonable to expect that any realization of  $H$ -flux on a torus would lead to similar difficulties.<sup>8</sup> We are therefore propose that the  $R$ -space does not exist as a low-energy effective notion decoupled from the stringy effects.

## 5. Comments on possible connections to the non-associative tori

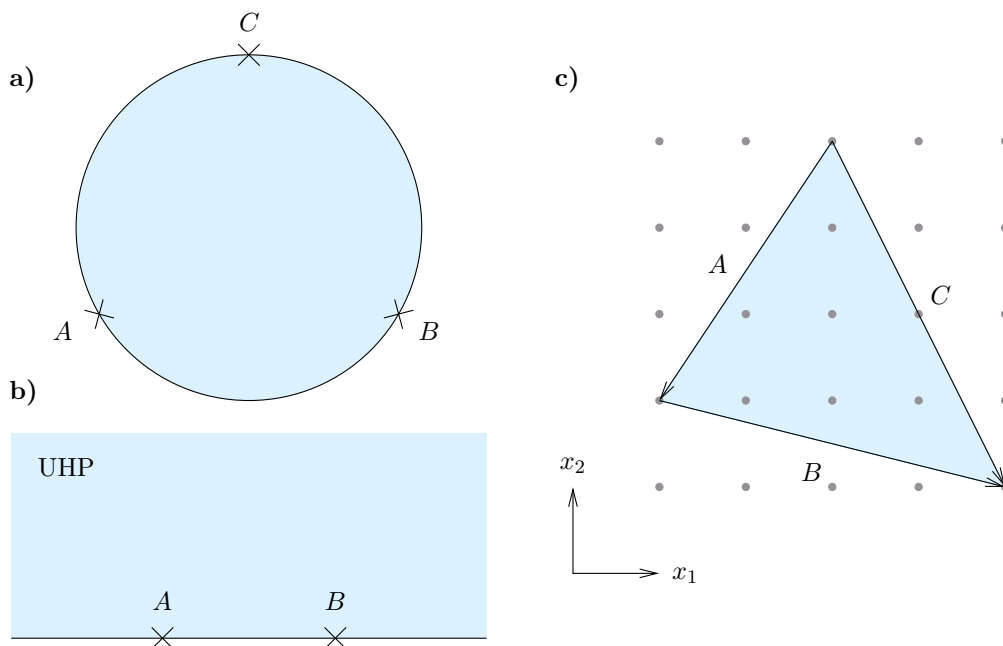
One of the main conclusions of this article is the observation that a sensible decoupling limit of low-energy open strings in  $R$ -space does not appear to exist. As we saw in the previous section, the main cause of this difficulty is the strong gravitational backreaction in the dual  $H$ -space description of the background geometry. It turns out, however, that an intriguing non-associative geometry, similar to the one described in [21], emerges if one naively ignores the gravitational backreaction. The fact that we are ignoring the gravitational backreaction, which is necessary in order to have a consistent background for string perturbation theory, makes the physical interpretation of this mathematical structure in terms of string theory less clear. Nonetheless, a connection to some kind of non-associative geometry as an effective description of  $R$ -space is sufficiently intriguing that we felt it worth illustrating. Our hope is that this discussion can be made more transparent in the future.

### 5.1 Review of dual lattice formulation of non-commutative geometry

We begin the discussion by recalling the approach used in [34–36] to describe ordinary non-commutative spaces. A D2-brane in a  $B$ -field background becomes a non-commutative gauge theory in a certain scaling limit. Instead of the D2-branes in a  $B$ -field background, however, one can also consider D0-branes in the T-dual torus of size  $L = \alpha'/\tilde{L}$ , which also has a non-vanishing  $B$ -field.

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<sup>8</sup>Recently, a large class of solutions of the four dimensional effective field theory, where the effects of the fluxes in compact dimensions are encoded in the superpotential, were constructed in [38]. It would be interesting to see if any of these constructions would allow a smooth decoupling limit to be taken along the lines discussed in this paper. To study this issue, however, it is essential to first find an explicit lift of these solutions to a solution of supergravity in ten dimensions similar to (2.9).



**Figure 2:** World sheet diagram corresponding to the scattering of open string winding modes ending on a periodic array of D-branes. Diagram a) shows the relevant disk geometry with three punctures representing the open strings. In b) the same geometry is shown in the upper half plane. In c) we show the space-time embedding of the saddle-point configuration for the constant  $B$  background. The gray circles represent the lattice of D0-branes. The punctures  $A$ ,  $B$ , and  $C$  are mapped to the edges of the triangle.

Before T-dualizing, the non-commutativity manifests itself in the three-point scattering amplitude of open string momentum modes on the D2:

$$A_{Non-comm}(p, q, r)(2\pi)^3 \delta^3(p + q + r) = e^{\frac{i}{2} p_1 \theta p_2} A_{Comm}(p, q, r)(2\pi)^3 \delta^3(p + q + r). \quad (5.1)$$

From the T-dual, D0-brane perspective, this is a scattering of winding modes. The configuration minimizing the world sheet action,

$$S = \frac{1}{4\pi\alpha'} \int g_{ij} dx^i \wedge *dx^j + B_{ij} dx^i \wedge dx^j, \quad (5.2)$$

can be visualized as a minimal area triangle. When the world sheet is parameterized by the upper half plane, this triangle corresponds to the configuration,

$$x_i = \frac{p_i}{2\pi i} \log\left(\frac{z}{\bar{z}}\right) + \frac{q_i}{2\pi i} \log\left(\frac{z-1}{\bar{z}-1}\right), \quad z \in H^+ \quad (5.3)$$

and is a solution to the equation of motion,

$$\nabla^2 x_i = 0, \quad (5.4)$$

which is independent of the  $B$ -field since  $dB = 0$ . However, the world sheet path integral picks up a  $B$ -dependent phase factor,

$$\exp\left[\frac{i}{4\pi\alpha'} \int B_{ij} dx^i \wedge dx^j\right] = \exp\left[\frac{i}{2} p_i \theta^{ij} q_j\right], \quad p_i = \frac{2\pi m_i}{\tilde{L}_i}, \quad q_i = \frac{2\pi n_i}{\tilde{L}_i}, \quad (5.5)$$

where

$$\theta = 2\pi\alpha' B . \tag{5.6}$$

This phase factor is equivalent to the flux of  $B$  through the triangle illustrated in figure 2 and allows one to define the Moyal product,

$$e^{ipx} * e^{iqx} = e^{ip\theta q/2} e^{i(p+q)x} . \tag{5.7}$$

Non-commutative gauge theory has a clean realization in string theory as the decoupling limit  $\alpha' \rightarrow 0$  with  $\tilde{L}_i$  fixed. In this limit, the masses of open string winding modes remain finite. Interpreting the winding modes as the momentum modes of the T-dual picture, one recovers precisely the dynamics of decoupled open strings in the Seiberg-Witten picture [12]. When the background  $B$ -field is scaled appropriately, the non-commutativity parameter as seen by the momentum modes in the dual picture also remains finite. This is how one reconstructs the effective physics of non-commutative gauge theories from the winding modes. Of course, in the case of a constant  $B$ -field background, one can T-dualize explicitly and obtain the same non-commutative field theory from either of the two approaches. In the context of the  $H/R$  duality, only one approach is available so we use the one to study the other.

### 5.2 Triple T-duality: $H \leftrightarrow R$

Several new feature arise when the  $H$ -field is non-vanishing. First, we must consider a three dimensional array of D-branes localized inside the  $H$ -space. The open string can now wind in three independent directions.

For the non-interacting strings, the mass of the wound strings are unaffected by the  $H$ -field. This is because one of the extended directions of a non-interacting string is the time component, whereas the  $B$ -field has no non-vanishing time-like component. Since the spectrum of the non-interacting wound strings are unaffected by the  $H$ -field, one concludes that the geometry of the theory must be encoded in the interaction terms.

When a string wound along  $(m_1, m_2, m_3)$  joins with a string wound along  $(n_1, n_2, n_3)$  to become a string which winds along  $(m_1 + n_1, m_2 + n_2, m_3 + n_3)$ , one expects to find a world sheet configuration which forms a triangle in a three dimensional lattice, as illustrated in figure 3.

If one naively follows the prescription of attributing the flux of  $B$ -field through this triangle as a phase, one obtains an expression for a generalization of the Moyal product. Letting

$$\vec{x} = (L_1 m_1, L_2 m_2 + L_3 m_3)(1 - \sigma_1) + (L_1 n_1, L_2 n_2, L_3 n_3)\sigma_2 \tag{5.8}$$

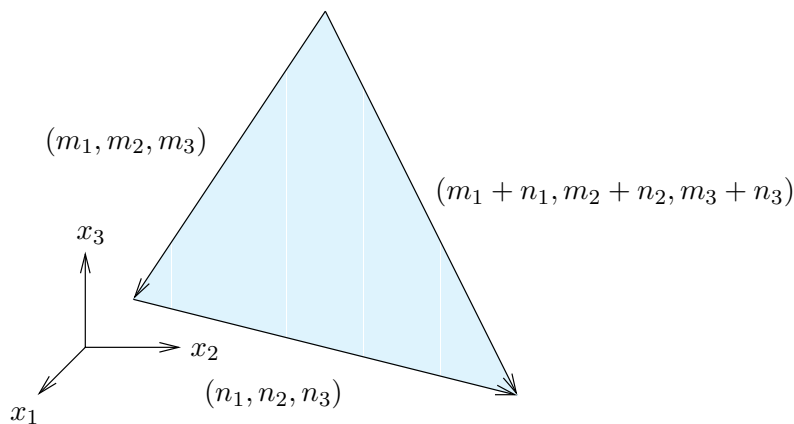
for  $0 < \sigma_1, \sigma_2 < 1$  and  $\sigma_1 + \sigma_2 < 1$ , one finds<sup>9</sup>

$$\pi(u(m, n)) \equiv e^{\frac{1}{4\pi\alpha'} \int B_{ij} dx^i \wedge dx^j} = e^{-2\pi i N((2m_1+n_1)/6+c)(m_2 n_3 - m_3 n_2)} . \tag{5.9}$$

The parameter  $c$  corresponds to the freedom to move the origin of the D0-brane lattice along  $x_1$ , but will not matter in most of the discussion. One can use this phase to define a new product,

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<sup>9</sup>We adopt a notation similar to the one in [21] to facilitate comparison of our algebra with theirs.



**Figure 3:** Minimal area triangle configuration for the open string winding modes ending on a three dimensional array of D-branes. The lattice points are suppressed for clarity.

$$e^{ipx} * e^{iqx} = \pi(u(m, n))e^{i(p+q)x}, \quad p_i = \frac{2\pi m_i}{\tilde{L}_i}, \quad q_i = \frac{2\pi n_i}{\tilde{L}_i}, \quad (5.10)$$

which can also be written in the form,

$$f(x) * g(x) = e^{-\frac{1}{(2\pi)^2} N \tilde{L}_1 \tilde{L}_2 \tilde{L}_3 \left( \frac{2\partial_{x_1} + \partial_{y_1}}{6} + C \right) (\partial_{x_2} \partial_{y_3} - \partial_{x_3} \partial_{y_2})} f(x)g(y) \Big|_{x=y}, \quad (5.11)$$

which acts on functions  $f(x)$  and  $g(x)$  in the  $R$ -space.<sup>10</sup>

An interesting novel feature of this product is that it is non-associative. Indeed, one can easily confirm that

$$(e^{ipx} * e^{iqx}) * e^{irx} = \pi(u(m, n))\pi(u(m + n, l))e^{i(p+q+r)x} \quad (5.12)$$

corresponding to the diagram on the left in figure 4, and

$$e^{ipx} * (e^{iqx} * e^{irx}) = \pi(u(m, n + l))\pi(\alpha_m(u(n, l)))e^{i(p+q+r)x}, \quad (5.13)$$

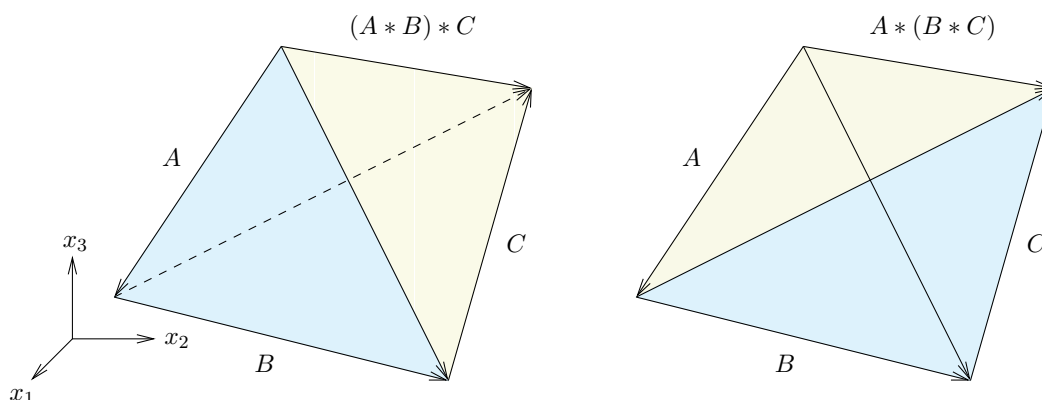
corresponding to the diagram on the right in figure 4, are not equal since

$$\phi(m, n, l) \equiv \frac{\pi(u(m, n + l))\pi(\alpha_m(u(n, l)))}{\pi(u(m, n))\pi(u(m + n, l))} = e^{\frac{i\pi N}{3}(m \cdot n \times l)} \neq 1. \quad (5.14)$$

Here,

$$\alpha_m \pi(u(n, l)) = e^{-2\pi i N ((2n_1 + l_1)/6 + m/2 + c)(n_2 l_3 - n_3 l_2)} \quad (5.15)$$

<sup>10</sup>We are using the fact that points on  $R$ -space can be viewed as an ordinary  $T^3$  when interactions are ignored.



**Figure 4:** Two different triangulations of quadrilateral representing the two different orders of multiplying the open strings  $A$ ,  $B$  and  $C$ . The difference of  $B$ -flux through the two triangulations is equivalent to the volume integral of the  $H$ -flux through the tetrahedron, and is a measure of non-associativity of the product (5.10).

accounts for the fact that  $\vec{n} + \vec{l}$  does not start at the tail of  $\vec{m}$ .<sup>11</sup>

The non-associativity can be understood in terms of two different ways of triangulating a quadrilateral, illustrated in figure 4. The difference in phases is the surface integral of  $B$  over the tetrahedron, which by Stokes theorem is equivalent to the volume integral of  $H$ . This is algebra appears to have the same form as the twisted crossed product considered in [21]. (Compare with equation (2.3) of [21]). This type of non-associative algebra has also appeared in other contexts [39] as well as in attempts to generalize Moyal algebra to higher dimensions in the context of branes<sup>12</sup> [40].

The picture illustrated in figure 4 also clarifies the fact that the non-associative product (5.10) is cyclic, i.e. the associator is a total derivative. In other words, we have

$$\int d^3x e^{ipx} * (e^{iqx} * e^{irx}) = \int d^3x (e^{ipx} * e^{iqx}) * e^{irx} = (2\pi)^3 \delta^3(p + q + r) . \quad (5.16)$$

This is because the conservation of momentum in the  $R$ -space constrains the tetrahedron to collapse into a triangle. Cyclicity is a useful notion in defining field theories using fields whose algebra is non-associative [41]. This means that one can unambiguously write an action whose interaction terms are at most cubic, such as  $\phi^3$  theory.<sup>13</sup>

The non-associative algebra being discussed here comes about somewhat differently from seemingly similar setup discussed in [43, 44]. These authors considered D3-branes in a non-trivial  $H$ -field background. Therefore, their program should be thought of as the

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<sup>11</sup>If  $N$  is even, however, the shift in the phase is a multiple of  $2\pi$  and does not affect the result. The simplicity of even  $N$  is related to the fact that Morita equivalence  $\Theta \rightarrow \Theta + 1$  is an isomorphism which acts non-trivially, while  $\Theta \rightarrow \Theta + 2$  acts trivially.

<sup>12</sup>We thank Y. Matsuo for telling us about their on-going work.

<sup>13</sup>A different approach for defining gauge theories with non-associative fields can be found in [42].



study of D3-branes in the  $H$ -space. What we consider instead is D3-branes in  $R$  space, or equivalently, D0-branes in  $H$ -space. The basic setup is therefore distinct, although certain aspects of the vertex operator algebra are inevitably similar. It should also be noted that a D3 in a presence of an  $H$ -field acquires an induced magnetic charge [45] via the Hanany-Witten mechanism [46, 47]. Since the  $H$ -space is compact, additional steps are needed to cancel this induced charge when constructing a consistent string theory background.

The algebra (5.10) is extremely similar in structure to the Busby-Smith algebra described in section 3 of [21]. At the present time, it is not completely clear how one should properly interpret the mathematical formalism described in [21] in physical terms using string theory. According to the authors of [21], the framework described in that paper does not directly concern D-branes (other than the fact that their charges are encoded by the relevant K-theory) and should be viewed as a statement regarding the closed strings. The fact that we identify similar algebraic structure in the lattice of dual branes appears to suggest that these structures are more natural in the context of open strings dynamics. It is also worth noting that [21] also describes the  $Q$ -space by “a continuous field of stabilized non-commutative tori” that is reminiscent of the description of  $Q$ -space in section 3, which is definitely an open string construction. It is an important open problem to clarify the proper physical interpretation of [21] and to settle the question of the relevance of the open v.s. closed strings.<sup>14</sup> What is needed is the analogue of [34].

Some of this discussion, however, is purely academic in that the story of the tetrahedron relied entirely on the triangular world sheet. Such a world sheet is a solution to the world sheet equation of motion (5.4) when  $H = 0$ , but in the case of non-vanishing  $H$ -field, (5.4) is corrected to

$$d * dx_i - H_{ijk} dx_j \wedge dx_k = 0 . \tag{5.17}$$

Interestingly, this equation of motion originally arose in an attempt to describe vortices in a superfluid [48, 49] and has resurfaced in various contexts [50–53]. The triangular world sheet (5.3) is not solved by this equation. It is not clear if there exist analytic solutions to this equation, nor is it clear how to use it to properly modify the naive product described above.

The fact that the non-associative product structure (5.10) is difficult to realize in a fully consistent treatment of critical string theory may be another indication that  $R$ -space does not exist as a low-energy effective notion of critical string theory. As a possible alternative, let us point out that there does exist a topological sigma model, known as the Poisson-WZ sigma model [54], which nicely reproduces our product. The action of the Poisson-WZ sigma model is given by

$$S = \int \eta_i \wedge dx^i + \frac{2\pi}{L_1 L_2 L_3} N x_1 dx_2 dx_3, \tag{5.18}$$

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<sup>14</sup>This need not be a mutually exclusive statement since branes can be transmuted into fluxes and vice versa. Nonetheless, it may turn out that the brane description is the most natural framework for providing a physical interpretation of [21].

where  $\eta_i$  is a world sheet one-form. Because of the invariance of the action under the gauge transformation,

$$\eta \rightarrow \eta + d\lambda, \tag{5.19}$$

one should consider the gauge fixed action,

$$S = \int \eta_i \wedge dx^i + \eta_i \wedge *d\gamma^i + \frac{2\pi}{L_1 L_2 L_3} N x_1 dx_2 dx_3 . \tag{5.20}$$

The fields  $\gamma$  and  $\eta$  constrain  $x$  to be harmonic, making (5.3) the unique solution given the boundary conditions. The Poisson-WZ model appears to play a role very similar to the Poisson sigma model [55, 56] whose boundary correlation functions were shown in [57] to elegantly reproduce the deformation quantization formula of Kontsevich [58]. A sophisticated interplay of ideas involving the Poisson sigma model and the Poisson-WZ model have been discussed, for example, in [59–61]. It is quite likely that the algebraic structure (5.10) and its connection to [62] will turn out to be most transparent in the context of the open Poisson-WZ sigma model winding modes in a manner closely resembling the discussion of section 5.2.

Unfortunately, Poisson-WZ sigma model, like the Poisson sigma model, are too general a construct to embed consistently in critical string theory. Not all consistent Kontsevich  $*$ -deformations are expected to be realizable as a decoupling limit of a consistent critical string construction. The absence of strong conceptual connection between decoupled open strings and the Poisson sigma model was also emphasized in [63]. This is in line with the our empirical observation that the consistent decoupling limit of effective field theory on  $R$ -space does not appear to exist.

## 6. Concluding remarks

In this article, we investigated the possible low-energy effective description of non-geometric compactifications discussed recently in the context of novel compactifications [1–3, 7]. We followed the guiding principle that generalized notions of geometry should have a concrete meaning when the intrinsic non-locality associated with the string scale is decoupled from scale relevant to geometry.

There are two examples of non-geometric compactifications that arise naturally in the context of T-duals of  $T^3$  with  $H$ -flux. In the case of  $Q$ -space, we found that the novel non-geometric features can be understood in terms of a non-commutative geometry compactified using a Morita duality. This non-commutative gauge theory has a singular moduli-space that is resolved by integrating in certain degrees of freedom. All of these degrees of freedom can be embedded in a UV complete theory given by little string theory coupled to flavor matter, but decoupled from gravity.

In the case of the  $R$ -space, we did not find a sensible decoupling limit. It is entirely possible that a triple T-duality of  $T^3$  with  $H$ -flux does not admit a low-energy effective description. Nonetheless, we did encounter certain features suggestive of a non-associative generalization of non-commutative geometry. That such a generalization to non-commutative geometry should arise in the  $R$ -space which is seemingly more “non-geometric” than the

$Q$ -space is sensible, and it would be extremely interesting if this connection can be made more precise.

The primary obstacle to understanding the non-associative structure from critical string theory is the strong gravitational back reaction. This problem appears to be evaded in the Poisson-WZ sigma model, which is a topological theory similar to the Poisson sigma model. Ultimately, we may find that  $R$ -space has a natural interpretation in terms of non-associative deformation of space-time only in the framework the topological sigma model and that this structure cannot be embedded into critical string theory.

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